UDC 536.25

CLASSICAL SOLUTIONS OF THE SECOND BOUNDARY VALUE PROBLEM FOR EQUATIONS OF UNSTEADY FREE CONVECTION

PMM Vol. 37, №1, 1973, pp. 184-190
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(Received June 14, 1971)

Substantiation is given for the use of the method of consecutive approximations to determine the velocity of the temperature distributions when unsteady free convection takes place in a closed convex region, at the surface of which the thermal flux density is specified. The domains of applicability of this method and the method of small parameter, coincide. The convergence of the approximate solution to the exact solution is proved and an estimate of the error is given. The local existence theorem and the uniqueness theorem are proved.

1. Let a convex region Ω be completely filled with an incompressible viscous fluid of constant initial temperature b_0. Let for $t_1 > 0$ a heat flux distribution $q_1(x_{1s}, t_1)$ be specified at the surface S of the vessel. We derive equations describing unsteady free convection in a bounded region with the boundary conditions of the second kind.

In the Bussinesq [1] approximation the equation of state assumes the form

$$\rho(b) = \rho(\langle b\rangle)\,[1 - \beta(\langle b\rangle)\,(b - \langle b\rangle)] \tag{1.1}$$

where ρ is the density, β the thermal expansion coefficient, b the temperature and $\langle b\rangle$ the volume-averaged temperature of the fluid. It was shown experimentally in [2] that

$$|\langle b\rangle - b_1| \ll |b - \langle b\rangle|$$

and from the formula for $\beta(b)$ in the Bussinesq approximation it follows that $\beta(\langle b\rangle) \simeq \beta(b_1)$

$$|\rho(\langle b\rangle) - \rho(b_1)| = \rho(b_1)\beta(b_1)\,|\langle b\rangle - b_1| \ll \beta(\langle b\rangle)\,\rho(b_1)\,|b - \langle b\rangle| \ll \rho(b_1)$$

where b_1 denotes the average temperature of the fluid. Consequently $\rho(\langle b\rangle) = \rho(b_1) = \rho_0$ and the equation of state assumes the form

$$\rho(b) = \rho_0[1 - \beta_0(b - \langle b\rangle)] \quad (\beta_0 = \beta(b_1)) \tag{1.2}$$

Transforming the continuity and heat transfer equations of Navier-Stokes in the usual manner employed in deriving the equations of steady free convection [1] and reducing them to the dimensionless form, we obtain the following system of equations as well as the initial and boundary conditions on S:

$$\frac{\partial \mathbf{V}}{\partial t} + (\mathbf{V}\cdot \operatorname{grad})\,\mathbf{V} = -\operatorname{grad} p - G\sigma^2 (T - \langle T\rangle)\,\mathbf{i} + \sigma\Delta\mathbf{V}$$

$$\operatorname{div}\mathbf{V} = 0, \qquad \partial T/\partial t + (\mathbf{V}\cdot\operatorname{grad} T) = \Delta T \tag{1.3}$$

$$\mathbf{V}|_{t=0} = 0, \qquad \mathbf{V}|_{\mathbf{x}_s} = 0, \qquad T|_{t=0} = 0, \qquad \left.\frac{\partial T}{\partial n}\right|_{\mathbf{x}_s} = q(\mathbf{x}_s, t) \tag{1.4}$$

Here

$$\mathbf{x} = \frac{\mathbf{x}_1}{l}, \quad t = \frac{a_0 t_1}{l^2}, \quad p = \frac{(P_1 - P_0)\, l^2}{\rho_0 a_0^2}, \quad \mathbf{V} = \frac{\mathbf{V} l}{a_0}$$

$$T = \frac{(b - b_0)\,\lambda_0}{q_0 l}, \quad \langle T \rangle = \frac{(\langle b \rangle - b_0)\,\lambda_0}{q_0 l} = \frac{Qt}{\Omega_0} \;[3,4]$$

$$\sigma = \frac{\nu_0}{a_0}, \quad G = \frac{g \beta_0 q_0 l^4}{\lambda_0 \nu_0^2}, \quad q = \frac{q_1}{q_0}, \quad q_0 = \max_{(\mathbf{x},\, t) \in S_t} q_1$$

$$\mathbf{x} \in \Omega, \quad \mathbf{x}_s \in S, \quad S_t = S \times [0, t], \quad Q = \int_S q\, ds$$

where l is the characteristic linear dimension, Ω_0 is the dimensionless volume, $\mathbf{V}$ is the velocity, P_1 and P_0 are the pressure and hydrostatic pressure, λ_0 is the thermal conductivity, a_0 is the thermal diffusivity and ν_0 is the kinematic viscosity. The above notation relates to the fluid under investigation.

The validity of Eqs. (1.3) is indirectly confirmed by the good agreement of the approximate solution of the initial boundary value problem (1.3), (1.4) with the experimental data obtained for a number of vessels in [3 – 5].

We shall call the functions $\mathbf{V}$, p and T the classical solution of the problem (1.3), (1.4) provided that these functions as well as the derivatives of T with respect to the spatial coordinates are continuous over the set of variables $\mathbf{x}$ and t in $\Omega_{t_0}^* = \Omega^* \times [0, t_0]$, have derivatives with respect to the set of variables $\mathbf{x}$ and t in $\Omega_{t_0}' = \Omega' \times [0, t_0]$, which are all continuous, appear in the system of equations (1.3) and satisfy the conditions (1.4) where $\Omega' \subset \Omega$ and Ω^* is the closure of region Ω.

2. Theorem 1. If Ω is a convex region, S is a twice continuously differentiable surface, $q(\mathbf{x}_s, t) \subset H^{1, 1/2}(S_{t_0})$ [6] and $q(\mathbf{x}_s, 0) = 0$, then a classical solution of the problem (1.3), (1.4) exists in Ω_{t_0} and can be obtained by the method of successive approximations, where $S_{t_0} = S \times [0, t_0]$.

Proof. Consider the sequences of functions $\mathbf{V}_k$, T_k and p_k satisfying the following systems of equations

$$\mathbf{V}_0 = 0, \quad p_0 = 0, \quad \frac{\partial T_0}{\partial t} = \Delta T_0$$

$$\frac{\partial \mathbf{V}_k}{\partial t} - \sigma \Delta \mathbf{V}_k = -\operatorname{grad} p_k - (\mathbf{V}_{k-1} \cdot \operatorname{grad})\, \mathbf{V}_{k-1} - G\sigma^2 (T_k - \langle T \rangle)\, \mathbf{i}, \quad \operatorname{div} \mathbf{V}_k = 0 \tag{2.1}$$

$$\frac{\partial T_k}{\partial t} = \Delta T_k - \frac{\partial}{\partial x_j} (V_{j,\, k-1} T_{k-1}) \qquad (k \geqslant 1)$$

with the initial and boundary conditions

$$T_k |_{t=0} = 0, \quad \left. \frac{\partial T_k}{\partial n} \right|_{\mathbf{x}_s} = q(\mathbf{x}_s, t), \quad \mathbf{V}_k |_{t=0} = 0, \quad \mathbf{V}_k |_{\mathbf{x}_s} = 0, \quad (k \geqslant 0) \tag{2.2}$$

Let us obtain the estimates for the functions $\mathbf{V}_k$ and T_k. We denote

$$U_k(t) = \max_{(i = 1,\, 2,\, 3)} \max_{\mathbf{x} \in \Omega^*} | V_{i,\, k} | \tag{2.3}$$

$$W_k(t) = \max_{\mathbf{x} \in \Omega^*} \left\{ \left| \frac{\partial T_k}{\partial x_1} \right|, \left| \frac{\partial T_k}{\partial x_2} \right|, \left| \frac{\partial T_k}{\partial x_3} \right|, | T_k | \right\}$$

Using the results of [7] we estimate the functions $U_k(t)$

$$U_{k+1} \leqslant C_1 G \sigma^2 \int_0^t W_k(\tau) \left[\frac{1}{(t - \tau)^\beta} + e^{C_0 (t - \tau)} \right] d\tau + C_1 G \sigma^2 \int_0^t \langle T(\tau) \rangle \left[\frac{1}{(t - \tau)^\beta} + e^{C_0 (t - \tau)} \right] d\tau +$$

$$C_2 \int_0^t U_k^2 \left[\frac{1}{(t-\tau)^\gamma} + e^{C_0(t-\tau)}\right] d\tau \qquad \left(\gamma = \frac{1+\beta}{2}\right) \tag{2.4}$$

Here β is an arbitrarily small positive number, C_0, C_1 and C_2 are constants depending only on the region Ω.

Using the a priori estimate given in [6], we estimate T_{k+1}

$$\| T_{k+1} \|^{(2)}_{\Omega_t} \leqslant C \left[U_k(t) \sum_{n=1}^{3} \max_{(\mathbf{x},\, t) \in \Omega_t^*} \left| \frac{\partial T_k}{\partial x_n} \right| + \| q \|^{(1)}_{S_t} \right] \tag{2.5}$$

Here $\Omega_t = \Omega \times [0, t]$ and C is a constant which depends on the region Ω and the surface S; definitions $\| T_{k+1} \|^{(2)}_{\Omega_t}$ and $\| q \|^{(1)}_{S_t}$ are given in [6]. From (2.3) and (2.5) follows

$$W_{k+1} \leqslant C\, [\, \| q \|^{(1)}_{S_t} + 3U_k W_k] \tag{2.6}$$

Consequently for $k \geqslant 0$ the functions $U_{k+1}(t)$ and $W_{k+1}(t)$ satisfy the inequalities

$$U^{(t)}_{k+1} \leqslant \int_0^t K_1(t-\tau)\, W_k(\tau)\, d\tau + \int_0^t K_2(t-\tau)\, U_k^2(\tau)\, d\tau + E_0(t), \qquad U_0 = 0 \tag{2.7}$$

$$W_{k+1}(t) \leqslant D_0(t) + 3CU_k(t)\, W_k(t), \; W_0(t) \leqslant D_0(t)$$

where

$$K_1(t-\tau) = C_1 G \sigma^2 \left[\frac{1}{(t-\tau)^\beta} + e^{C_0 (t-\tau)} \right]$$

$$K_2(t-\tau) = C_2 \left[\frac{1}{(t-\tau)^\gamma} + e^{C_0 (t-\tau)} \right]$$

$$E_0(t) = \int_0^t K_1(t-\tau) \langle T(\tau) \rangle\, d\tau, \qquad D_0(t) = C \| q \|^{(1)}_{S_t}$$

Let us set

$$D_1(t) = 4E_0(t) + 4 \int_0^t K_1(t-\tau)\, D_0(\tau)\, d\tau$$

and denote by t_0 the least interval of time for which the unequalities

$$\int_0^t K_2(t-\tau)\, D_1^2(\tau)\, d\tau < \frac{1}{2} D_1(t), \qquad \int_0^t K_2(\tau)\, d\tau + 6C \left[\left(\frac{1}{12} - \frac{1}{6C} \int_0^t K_2(\tau)\, d\tau \right)^2 + \right.$$

$$\left. 2CD_0(t) \int_0^t K_1(\tau)\, d\tau \right]^{1/2} < \frac{11}{12} C, \quad 6CD_1(t) < 1 \tag{2.8}$$

hold.

Using the method of mathematical induction we can show that for any $k \geqslant 0$ the following inequalities hold for $t \in [0, t_0]$:

$$U_k(t) < D_1(t), \quad W_k(t) < 2D_0(t) \tag{2.9}$$

Consequently T_k, $\mathbf{V}_k$ and $\partial T_k / \partial x_j$ are functions which are uniformly bounded for $t \in [0, t_0]$ $(j = 1, 2, 3)$.

We shall show that the sequences $\mathbf{V}_k$, T_k and $\partial T_k / \partial x_j$ $(j = 1, 2, 3)$ converge as $k \to \infty$. Using (2.1) and (2.2) we obtain a system of eqautions which are satisfied by the functions $\mathbf{V}_{n+1} - \mathbf{V}_n$ and $T_{n+1} - T_n$ with the zero initial and boundary conditions. Let us set

$$P_{n+1} = \max_{(i=1,\,2,\,3)} \; \max_{(\mathbf{x},\, t) \in \Omega_{t_0}^*} | V_{n+1,\, i} - V_{n,\, i} |$$

$$q_{n+1} = \max_{(\mathbf{x},\,t)\in\Omega_{t_0}} \left\{ |T_{n+1} - T_n|, \left| \frac{\partial (T_{n+1} - T_n)}{\partial x_i} \right| \right\} \qquad (i = 1, 2, 3)$$

Using the fact that T_k and $\mathbf{V}_k$ are bounded uniformly, we obtain

$$P_{n+1} \leqslant \varsigma_n \int_0^t K_1(\tau)\, d\tau + \frac{P_n}{3C} \int_0^t K_2(\tau)\, d\tau$$

$$q_{n+1} \leqslant \frac{q_n}{6} + 2CD_0 P_n \tag{2.10}$$

Let us denote

$$\mathbf{X}_n = \left\| \begin{matrix} P_n \\ \\ q_n \end{matrix} \right\|, \quad \mathbf{A} = \left\| \begin{matrix} \frac{1}{3C} \int_0^t K_2(\tau)\, d\tau & \int_0^t K_1(\tau)\, d\tau \\ 2CD_0 & {}^1/_6 \end{matrix} \right\|$$

Then (2.10) reduces to

$$\mathbf{X}_{n+1} \leqslant \mathbf{A}\mathbf{X}_n \tag{2.11}$$

Using [8] we obtain from (2.11)

$$|\mathbf{X}_{n+1}| \leqslant |\mathbf{A}\mathbf{X}_n| \leqslant \|\mathbf{A}\|\, |\mathbf{X}_n| \leqslant \lambda_{\max} |\mathbf{X}_n| \tag{2.12}$$

Here $\lambda_{\max}$ is the largest eigenvalue of the matrix $\mathbf{A}$ and $|\mathbf{X}_n| = \sqrt{P_n^2 + q_n^2}$. By (2.8) $\lambda_{\max} < 1$. From (2.12) we obtain

$$|\mathbf{X}_n| \leqslant \lambda_{\max}^n |\mathbf{X}_0|$$

$$P_n \leqslant \lambda_{\max}^n |\mathbf{X}_0|, \quad q_n \leqslant \lambda_{\max}^n |\mathbf{X}_0|$$

Since $\lim_{n\to\infty} \lambda_{\max}^n = 0$, we find that $\lim P_n = 0$ and $\lim q_n = 0$ as $n \to \infty$.

Consequently the sequences $\mathbf{V}_n$, T_n and $\partial T_n / \partial x_j$ represent the fundamental sequences of the functions. Therefore, since $\mathbf{V}_n$, T_n and $\partial T_n / \partial x_j$ are uniformly bounded functions, the following limits exist when $n \to \infty$

$$\lim \mathbf{V}_n = \mathbf{V}', \quad \lim T_n = T', \quad \lim \frac{\partial T_n}{\partial x_j} = T_j'$$

Using [7, 9] it can be shown that $\mathbf{V}_n$ and T_n represent the classical solution of the problem (2.1), (2.2).

Since $\mathbf{V}_k$, T_k and $\partial T_k / \partial x_m$ are continuous functions over the variables $\mathbf{x}$, t up to the boundary of the region Ω_{t_0} and converge uniformly to $\mathbf{V}'$, T' and T_m' when $k \to \infty$, then $\mathbf{V}'$, T' and T_m' are continuous functions over the variables $\mathbf{x}$ and t up to the boundary of Ω_{t_0} and assume the initial and boundary values on S in a continuous manner.

We shall show that the functions $\mathbf{V}'$ and T' represent a solution of the problem (1.3), (1.4). Consider the following system of equations:

$$\frac{\partial \mathbf{V}}{\partial t} - \sigma \Delta \mathbf{V} = -\frac{\partial}{\partial x_s} (\mathbf{V}' \mathbf{V}_s') - G\sigma^2 (T - \langle T \rangle)\, \mathbf{i} - \operatorname{grad} p \tag{2.13}$$

$$\operatorname{div} \mathbf{V} = 0, \quad \frac{\partial T}{\partial t} - \Delta T = -\frac{\partial}{\partial x_s} (V_s' T')$$

with the initial and boundary conditions on S

$$\mathbf{V}|_{t=0} = 0, \quad \mathbf{V}|_{\mathbf{x}_s} = 0, \quad T|_{t=0} = 0, \quad \left.\frac{\partial T}{\partial n}\right|_{\mathbf{x}_s} = q(\mathbf{x}_s, t)$$

Making use of the fact that $\mathbf{V}'$ and T' are bounded and applying the method employed in obtaining the estimates for $\mathbf{V}_k$ and T_k, we can show that

$$|V_i| < D_1, \quad |T| < 2D_0, \quad \left|\frac{\partial T}{\partial x_i}\right| < 2D_0 \quad (i = 1, 2, 3)$$

Subtracting (2.13) from (2.1) we obtain a system of equations satisfied by the functions $\mathbf{V}_{n+1} - \mathbf{V}$ and $T_{n+1} - T$. Obtaining for

$$\max_{(i=1,2,3)} \max_{(\mathbf{x},t)\in\Omega_{t_0}^*} |V_{n+1,i} - V_i|, \max_{(\mathbf{x},t)\in\Omega_{t_0}^*} \left\{|T_{n+1} - T|, \left|\frac{\partial (T_{n+1} - T)}{\partial x_m}\right|\right\}$$
$$(m = 1, 2, 3)$$

the estimates analogous to those obtained for P_{n+1} and q_{n+1}, making use of the facts that $\mathbf{V}_n$, $\mathbf{V}'$, T_n and T' are bounded and that $\lim V_n = V'$ and $\lim T_n = T'$ as $n \to \infty$, and performing in the inequality analogous to (2.11) the passage to the limit as $n \to \infty$, we arrive at

$$\lim_{n\to\infty} \max_{(i=1,2,3)} \max_{(\mathbf{x},t)\in\Omega_{t_0}^*} |V_i - V_{n+1,i}| = 0$$

$$\lim_{n\to\infty} \max_{(\mathbf{x},t)\in\Omega_{t_0}^*} \left\{|T_{n+1} - T|, \left|\frac{\partial (T_{n+1} - T)}{\partial x_m}\right|\right\} = 0$$

Consequently we find that as $n \to \infty$,

$$\mathbf{V}' = \lim \mathbf{V}_n = \mathbf{V}, \quad T' = \lim T_n = T \;\; (T_j' = \partial T / \partial x_j)$$

Therefore the functions $\mathbf{V}$ and T satisfy the system (1.3) and the initial and boundary conditions (1.4) on S in a continuous manner. This implies that $\mathbf{V}$ and T represent the classical solution of the problem (1.3), (1.4) in Ω_{t_0}, and the a priori estimates

$$|V_i| < D_1, \quad |T| < 2D_0, \quad \left|\frac{\partial T}{\partial x_i}\right| < 2D_0 \quad (i = 1, 2, 3)$$

hold in Ω_{t_0}.

Since $\mathbf{V}$, T and grad p satisfy the system (1.3) in Ω_{t_0}', then grad p is a function continuous in $\mathbf{x}$ and t in Ω_{t_0}'. Consequently the pressure p is also a function continuous in $\mathbf{x}$ and t in Ω_{t_0}'. Using the property of continuity of the functions p and grad p in Ω_{t_0}' we can show that the pressure p is a continuous function in $\Omega_{t_0}^*$.

Thus we have proved the theorem of existence of a classical solution of the problem (1.3), (1.4) local with respect to time, and shown an approximate method of obtaining this solution. The method consists of solving the system of linear equations (2.1) up to a certain value of k, and approximating the true solutions with the approximate ones.

Let us now estimate the rate at which the approximate solution converges to the exact solution. To do this we subtract from Eqs. (2.1) the corresponding Eqs. (1.3) and denote

$$\max_{(i=1,2,3)} \max_{(\mathbf{x},t)\in\Omega_{t_0}^*} |V_i - V_{k,i}| = E_k, \quad \Phi_k = \begin{Vmatrix} E_k \\ F_k \end{Vmatrix}$$

$$\max_{(\mathbf{x},t)\in\Omega_{t_0}^*} \left\{|T - T_k|, \left|\frac{\partial (T - T_k)}{\partial x_m}\right|\right\} = F_k \quad (m = 1, 2, 3)$$

Performing estimates analogous to (2.10) we find, that $\Phi_{k+1} \leqslant A\Phi_k$. Solving this inequality as we solved (2.11), we obtain

$$|\Phi_k| \leqslant \lambda_{\max}^k |\Phi_0|, \qquad t \in [0, t_0]$$

which yield the following estimate of the error:

$$|V_j - V_{j,k}| \leqslant 2 \max_{t\in[0,t_0]} \lambda^k_{\max} \max_{t\in[0,t_0]} \{D_1, 2D_0\}$$
$$(j = 1, 2, 3)$$
$$|T - T_k| \leqslant 2 \max_{t\in[0,t_0]} \lambda^k_{\max} \max_{t\in[0,t_0]} \{D_1, 2D_0\}$$

Note. From Theorem 1 and the formulas (2.8) it follows that the method of consecutive approximations can be used not only for computing week unsteady free convection ($t_0 \gg 1$, $R = G\sigma \ll 1$), but also to compute a developed unsteady free convection ($t_0 \ll 1$, $R \gg 1$). Thus the domains of applicability of the method of successive approximation and the method of small parameter (R is a small parameter) coincide, although the former is simpler for computations.

3. Theorem 2. The classical solution of the problem (1.3), (1.4) is unique.

Proof. Assume that two solutions ($\mathbf{V}, T, p$) and ($\mathbf{V}', T', p'$) exist. We denote $\mathbf{W} = \mathbf{V} - \mathbf{V}'$, $\tau = T - T'$ and $P = p - p'$. Then, using (1.3), (1.4) we obtain the following system of equations for $\mathbf{W}$, τ and P:

$$\frac{\partial \mathbf{W}}{\partial t} - \sigma\Delta\mathbf{W} = -\frac{\partial}{\partial x_k}(V_k\mathbf{W} + \mathbf{V}'W_k) - \operatorname{grad} P - G\sigma^2\tau\mathbf{i}, \tag{3.1}$$

$$\operatorname{div} \mathbf{W} = 0 \tag{3.2}$$

$$\frac{\partial \tau}{\partial t} - \Delta\tau = -\frac{\partial}{\partial x_k}(V_k\tau + TW_k) \tag{3.3}$$

with homogeneous initial and boundary conditions. Multiplying (3.1) by $\mathbf{W}$ and (3.3) by τ and integrating the resulting expressions over Ω we find, with the help of (3.2) and the boundary conditions, the following respective expressions

$$\frac{d}{dt}\int_\Omega \mathbf{W}^2 d\Omega + 2\sigma \int_\Omega \sum_{s=1}^{3} \left(\frac{\partial \mathbf{W}}{\partial x_s}\right)^2 d\Omega = 2\int_\Omega \mathbf{V}'W_k\mathbf{W}_{x_k} d\Omega - 2\sigma^2 G \int_\Omega \tau(\mathbf{W}, \mathbf{i})\, d\Omega \tag{3.4}$$

$$\frac{d}{dt}\int_\Omega \tau^2 d\Omega + 2\int_\Omega \sum_{k=1}^{3} \left(\frac{\partial \tau}{\partial x_k}\right)^2 d\Omega = \int_\Omega \tau_{x_k} TW_k d\Omega \tag{3.5}$$

Estimating the integrals appearing in the right-hand sides of (3.4) and (3.5), using the fact that $\mathbf{V}'$ and T are bounded and the Cauchy-Buniakowski inequalities for the sums and integrals, and substituting the estimates obtained into (3.4) and (3.5), respectively, we obtain

$$\frac{d}{dt}\int_\Omega \mathbf{W}^2 d\Omega + (2\sigma - C\varepsilon)\int_\Omega \sum_{s=1}^{3} \left(\frac{\partial \mathbf{W}}{\partial x_s}\right)^2 d\Omega \leqslant \frac{C}{\varepsilon}\int_\Omega \mathbf{W}^2 d\Omega + G\sigma^2 \int_\Omega (\tau^2 + \mathbf{W}^2)\, d\Omega \tag{3.6}$$

$$\frac{d}{dt}\int_\Omega \tau^2 d\Omega + (2 - \varepsilon_1 C_1)\int_\Omega \sum_{k=1}^{3} \left(\frac{\partial \tau}{\partial x_k}\right)^2 d\Omega \leqslant \frac{C_1}{\varepsilon_1}\int_\Omega \mathbf{W}^2 d\Omega \quad (\varepsilon > 0,\ \varepsilon_1 > 0) \tag{3.7}$$

Here C is a constant depending on D_1, Ω and ε, while C_1 is a constant depending on D_0, Ω and ε_1. Taking $\varepsilon < 2\sigma / C$ and $\varepsilon_1 < 2 / C_1$ we obtain from (3.6) and (3.7) the following inequalities:

$$\frac{d}{dt}\int_\Omega \mathbf{W}^2 d\Omega \leqslant \left(\frac{C}{\varepsilon} + G\sigma^2\right)\int_\Omega \mathbf{W}^2 d\Omega + G\sigma^2 \int_\Omega \tau^2 d\Omega, \qquad \frac{d}{dt}\int_\Omega \tau^2 d\Omega \leqslant \frac{C_1}{\varepsilon_1}\int_\Omega \mathbf{W}^2 d\Omega$$

which combined together yield

$$dy / dt \leqslant ky$$

$$y(t) = \int_{\Omega} (W^2 + \tau^2)\, d\Omega, \qquad k = \frac{C}{\varepsilon} + \frac{C_1}{\varepsilon_1} + G\sigma^2 \tag{3.8}$$

Solving (3.8) under the condition that $y(0) = 0$, we find that $y(t) \leqslant y(0)\, e^{-kt} = 0$. Consequently $y(t) \equiv 0$ for any $t > 0$, which means that $V = V'$, $T = T'$ and $p = p' + \text{const}$.

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Translated by L.K.